Wieler Solenoids from Flat Manifolds Part 2: The Unstable Groupoid

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Acknowledgments

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Flat Manifold Inverse Limits

Let Y be a flat d-manifold and g a locally expansive n-fold cover. Let $(X, \varphi) = \varprojlim (Y, g)$, i.e.

$$X = \{(y_0, y_1, \dots) | y_i \in Y, g(y_{i+1}) = y_i\}$$

$$\varphi((y_0, y_1, y_2, \dots)) = (g(y_0), g(y_1), g(y_2), \dots) = (g(y_0), y_0, y_1, \dots)$$

$$\varphi^{-1}(y_0, y_1, y_2, \dots) = (y_1, y_2, \dots)$$

Unstable Equivalence

Definition

Given $x, y \in X$, we say $x \sim_u y$ are **unstably equivalent** if $\lim_{n\to\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0$

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- The "only if" direction is the result of the local expansiveness of g.
- Intuitively, we must eventually "choose the same pre-image."

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Definition (Unstable Groupoid)

$$\mathcal{G}_u = \{(x, y) | x \sim_u y, x, y \in X^s((z, z, z, \ldots), \varepsilon)\}$$

where z is the fixed point of g.

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- Then,

$$\mathcal{G}_u(X, \varphi) = \{(x, y) | x \sim_u y, x, y \in X^s((1, 1, 1, \ldots), \varepsilon)\}$$

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- Given $(x, y) \in X^s((1, 1, \ldots), \varepsilon) \times X^s((1, 1, 1, \ldots), \varepsilon), x \sim_u y$ if and only if for each $\varepsilon > 0$, there is an N such that for all $n \geq N, d(x_n, y_n) < \varepsilon$.

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- We can also view $Y = \mathbf{S}^1 \cong \mathbb{R}/\mathbb{Z}, \; g : z \mapsto 2z$
- Recall the full 2-shift $(\Sigma_{\{0,1\}}, \sigma)$ of two-sided binary sequences with the map: $\sigma((x_n))_k = x_{k+1}$
- We can characterize $\mathcal{G}_u$ using the Markov partition,

$$\pi : (\Sigma_{\{0,1\}}, \sigma) \to \varprojlim(\mathbb{R}/\mathbb{Z}, z \mapsto 2z)$$

$$\pi(\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots) = (.x_{-1}x_{-2}\ldots, .x_0x_{-1}x_{-2}\ldots, .x_1x_0x_{-1}x_{-2}\ldots)$$

- If $x = (0, .z_0, .z_1z_0, .z_2z_1z_0, \ldots)$ and $y = (0, .w_0, .w_1w_0, .w_2w_1w_0, \ldots)$ represented in binary then $x \sim_u y$ if and only either:
  1. For some $K, z_k = w_k$ for all $k \geq K$.
  2. For some $K, z_k(w_k) = 0$ for all $k \geq K$ and $w_k(z_k) = 1$, for all $k \geq K$.

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Motivating Example

- If $Y = \mathbf{S}^1$, $G = \mathbb{Z}$, and $g(z) = z^2$, then $g_*(\mathbb{Z}) = 2\mathbb{Z}, = g_*^j(\mathbb{Z}) = 2^j\mathbb{Z}$, and $G/G_j = \mathbb{Z}/2^j\mathbb{Z}$.

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- The associated odometer is:

$$\mathbb{Z} \curvearrowright (\varprojlim(\mathbb{Z}/2^j\mathbb{Z}, \text{coset inclusion})$$

$$= \mathbb{Z} \curvearrowright (\mathbb{Z}/\mathbb{Z} \leftarrow \mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z}/4\mathbb{Z} \leftarrow \mathbb{Z}/8\mathbb{Z} \leftarrow \cdots \cong \mathbb{Z}_2).$$

Example cont

- Representing the cosets in binary, we have:

$$\mathbb{Z}/2\mathbb{Z} = \{0, 1\},\ \mathbb{Z}/4\mathbb{Z} = \{00, 10, 01, 11\},$$

$$\mathbb{Z}/8\mathbb{Z} = \{000, 100, 010, 110, \ldots\}\text{etc.},$$

so that

$$\mathbb{Z}_2 = (x_0, x_0x_1, x_0x_1x_2, \ldots)$$

for $x_i \in \{0, 1\}$ and the generator $1 \in \pi_1(Y) \cong \mathbb{Z}$ acts on $\mathbb{Z}_2$ by

$$1 \cdot (x_0 + 2\mathbb{Z}, x_0x_1 + 4\mathbb{Z}, x_0x_1x_2 + 8\mathbb{Z}, \ldots) =$$

$$(1 + x_0 + 2\mathbb{Z}, 1 + x_0x_1 + 4\mathbb{Z}, 1 + x_0x_1x_2 + 8\mathbb{Z}, \ldots)$$

General Case

- A flat manifold Y with a locally expansive n-fold cover g determines two dynamical systems:
  1. The **Wieler Solenoid** $\varprojlim(\mathbf{Y}, \mathbf{g})$ with groupoid $\mathcal{G}_{\mathbf{u}}$
  2. The **odometer** $\pi_{\mathbf{1}}(\mathbf{Y}) \curvearrowright \varprojlim\left(\pi_{\mathbf{1}}(\mathbf{Y})/\mathbf{g}_*^{\mathbf{j}}(\pi_{\mathbf{1}}(\mathbf{Y}), \mathbf{i}_{\mathbf{j}}^{\mathbf{j}+1}\right)$ with groupoid $\mathcal{G}_{\mathbf{orbit}}$

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$$\pi : (\{0, 1, \ldots, n-1\}^{\mathbb{Z}} \cong (G/G_1)^{\mathbb{Z}}, \sigma) \to \varprojlim(Y, g)$$

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Upshot for Homology

Theorem (Scarparo(special case))

For Y flat, g a locally expansive $n-$fold cover, and $\pi_1(Y) \curvearrowright \varprojlim \left(\pi_1(Y)/g_*^j(\pi_1(Y)), i_j^{j+1}\right)$ the associated odometer,

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where $g!$ is the transfer map on homology.

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Theorem

For Y an oriented flat d-manifold,

$$H_*(\mathcal{G}^u) \cong H^{d-*}(X)$$

- Note: Doesn't hold when Y is not orientable

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Theorem (Scarparo, Baum-Connes, Carrión, Green, etc.)

Given Y flat, g a locally expansive $d-$fold cover, and $\pi_1(Y) \curvearrowright \varprojlim \left(\pi_1(Y)/g_*^j(\pi_1(Y)), i_j^{j+1}\right)$ the associated odometer,

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- Then
$$K_*(C_r^*(\mathcal{G}^u)) \cong K_*(C_r^*(\mathcal{G}_{orbit})) \cong \varinjlim(K_*(Y), g!)$$

- If Y is spin^c :
$$\varinjlim(K_*(Y), g!) \cong \varinjlim(K^{*+d}(Y), g^*),$$
where g^* is the induced map on K-theory

Upshot for K-Theory

Theorem (Scarparo, Baum-Connes, Carrión, Green, etc.)

Given Y flat, g a locally expansive $d-$fold cover, and $\pi_1(Y) \curvearrowright \varprojlim \left(\pi_1(Y)/g_*^j(\pi_1(Y)), i_j^{j+1}\right)$ the associated odometer,

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- Then
$$K_*(C_r^*(\mathcal{G}^u)) \cong K_*(C_r^*(\mathcal{G}_{orbit})) \cong \varinjlim(K_*(Y), g!)$$

- If Y is spin^c :
$$\varinjlim(K_*(Y), g!) \cong \varinjlim(K^{*+d}(Y), g^*),$$
where g^* is the induced map on K-theory
- Since $X = \varprojlim(Y, g)$:
$$K^{*+d}(X) \cong \varinjlim(K^{*+d}(Y), g^*).$$

K-theory cont

Theorem

For Y a spin^c flat n-manifold,

$$K_*(C_r^*(\mathcal{G}^u)) \cong K^{*+d}(X)$$

Doesn't hold in general when Y is not spin^c